

HIERARCHICAL MULTI-DIMENSIONAL LIMITING PROCESS ON CORRECTION VIA RECONSTRUCTION FOR COMPRESSIBLE EULER EQUATIONS

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ABSTRACT

The present paper deals with the continuous work of extending multi-dimensional limiting process (MLP) onto correction procedure via reconstruction (CPR). MLP, which has been originally developed in finite volume method (FVM), provides an accurate, robust and efficient oscillation-control mechanism in multiple dimensions for linear reconstruction. Recently, MLP has been extended into higher-order higher-order reconstruction. The proposed method, called hierarchical MLP, is developed in discontinuous Galerkin method, and it can be readily extended to CPR framework for solving compressible Euler equations. Through extensive numerical experiments, it is observed that that the proposed approach yields outstanding performances in resolving non-compressive as well as compressive flow features.

INTRODUCTION

In the last decade, various higher-order discretization methods have been developed to resolve complex flow structure more accurately. These methods owe merits of both finite volume methods (FVM) and finite elements methods (FEM). Thus, it becomes possible to develop higher-order reconstruction for each cell within minimum stencil and to evolve hyperbolic conservation laws, including Euler equations. Recently, Huynn and Wang propose correction procedures via reconstruction (CPR) [1,2], which provides unifying framework of well-developed higher-order methods, such as spectral volume/difference and discontinuous Galerkin (DG) methods. CPR shares many merits of these methods and it is more simple and efficient.

However, there are a few obstacles to extend higher-order methods, including CPR, in general high speed unsteady flows. One of them is to design a robust and efficient shock capturing algorithm to suppress unwanted oscillations around discontinuities without compromising the higher-order nature. The main reason for such oscillations is the lack of a proper diffusion mechanism, especially for multi-dimensional flows.

Recently, the multi-dimensional limiting process (MLP), with the theoretical foundation of the maximum principle, has been successfully proposed in the FVM. Compared with traditional limiting strategies, such as TVD or ENO-type schemes, MLP effectively controls unwanted oscillations particularly in multiple dimensions. A series of researches [3] clearly demonstrates that the MLP limiting strategy possesses superior characteristics in terms of accuracy, robustness and efficiency in computations on structured and unstructured grids in FVM.

In this study, we develop a new robust and accurate limiting algorithm for CPR, extending the MLP limiting philosophy to higher-order reconstruction. This approach, called the hierarchical MLP methods, are able to accurately capture complex compressible multi-dimensional flow structure without yielding unwanted oscillations. Since the proposed limiting algorithm relies only on the MLP stencil regardless of the order of reconstruction, it facilitates an easy extension to higher-order reconstruction. This limiting algorithm have been developed on DG framework [4], and it can be readily combined into CPR framework.

CORRECTION PROCEDURES VIA RECONSTRUCTION

CPR starts from the strong form of the governing equations.

$$\int_{T_j} \left[\frac{\partial \mathbf{Q}_j^h}{\partial t} + \nabla_h \cdot \mathbf{F}(\mathbf{Q}_j^h) + \delta_j \right] \mathbf{W} dV = 0 \quad (1)$$

where \mathbf{Q} is the conservative variable vector, \mathbf{F} are the flux. δ_j is the lifting operator.

$$\int_{T_j} \delta_j \mathbf{W} dV = \int_{\partial T_j} \left(\mathbf{H}_c(\mathbf{Q}_{jk}^h, \mathbf{Q}_{kj}^h) - \mathbf{F}(\mathbf{Q}_{jk}^h) \right) \cdot \mathbf{n} \mathbf{W} dS \quad (2)$$

where \mathbf{Q}_{jk}^h is the cell interface state vector in the direction from T_j to its neighboring cell T_k , and $\mathbf{H}(\mathbf{Q}_L, \mathbf{Q}_R)$ is the tensors of a numerical flux function. Depending on the definition of lifting operator, CPR can recover spectral volume/difference or DG formulations.

In CPR method, there are solution points to represent higher-order reconstruction on each cell. By imposing Eq. (1) on each solution point and projecting these onto V^n , we can obtain final formulation.

$$\frac{\partial \mathbf{Q}_{i,j}^h}{\partial t} + \Pi \left(\nabla_h \cdot \mathbf{F}(\mathbf{Q}_{i,j}^h) \right) + \delta_{i,j} = 0. \quad (3)$$

where Π is the projection operator. The details of CPR is included in Ref. [2].

HIERARCHICAL MLP LIMITING FOR CPR

The proposed limiting strategy starts from the MLP condition, which is an extension of the one-dimensional monotonic condition. The basic idea of the MLP condition is to control the distribution of both cell-centered and cell-vertex physical properties to mimic the multi-dimensional nature of flow physics.

$$\bar{q}_{v_i}^{\min} \leq q_{v_i} \leq \bar{q}_{v_i}^{\max} \quad (4)$$

where q is a state variable and q_{v_i} is the vertex value. $(\bar{q}_{v_i}^{\min}, \bar{q}_{v_i}^{\max})$ are the minimum and maximum of the cell-averaged values among all neighboring cells sharing the same vertex v_i . For a linear reconstruction, the MLP-u slope limiters satisfy the maximum principle on the MLP stencil to ensure multi-dimensional monotonicity while maintaining the second order of accuracy on unstructured grids [3].

For higher-order reconstruction, MLP condition is not sufficient to control spurious oscillation. Unlike linear reconstruction, controlling the value on the vertex only does not guarantee the monotonic distribution of the cell. Thus, MLP condition may not identify the troubled-cells which may lead to violation of the maximum principle. After some analysis, a more strict condition, called augmented MLP condition, is proposed to identify maximum-principle violating

Table 1 Grid refinement test for compressible inviscid flow with a sinusoidal source at $t = 1.0$.

	Grid	L^∞	Order	L^2	Order
	$h_0/20$	3.1456E-03		3.0090E-04	
Hierarchical	$h_0/40$	3.3802E-04	3.22	3.2533E-05	3.21
CPR- $P2$ -MLP	$h_0/80$	4.2948E-05	2.97	3.9841E-06	3.02
	$h_0/160$	6.2567E-06	2.79	5.0343E-07	3.00
	$h_0/20$	1.9468E-04		1.3410E-05	
Hierarchical	$h_0/40$	1.1861E-05	4.04	6.6620E-07	4.33
CPR- $P3$ -MLP	$h_0/80$	6.3917E-07	4.21	3.6941E-08	4.17
	$h_0/160$	4.6466E-08	3.80	2.2764E-09	4.03

cell.

$$\bar{q}_{v_i}^{\min} \leq q_{v_i}^{h,\min} \leq q_{v_i} \leq q_{v_i}^{h,\max} \leq \bar{q}_{v_i}^{\max}. \quad (5)$$

Based on the MLP condition and augmented MLP condition, we proposed hierarchical MLP method which consist of two step: At first, the troubled-cells are identified and the augmented MLP condition with extrema detector is used. For these troubled cells, the projection to V^{n-1} function space or MLP-u slope limiter is applied. This procedure is applied in a hierarchical manner from higher-order mode to $P2$ mode. Although this approach is originally proposed for scalar hyperbolic conservation laws, it can be applicable to solve compressible Euler equation.

NUMERICAL RESULTS

Compressible Flow with a Sinusoidal Source

We examine the grid convergence behavior of the proposed limiting for compressible flow with a source term. The reference analytic solution is a sinusoidal function and the corresponding source term can be obtained by inserting it.

$$\begin{aligned} \mathbf{Q}_t + \nabla \cdot \mathbf{F} &= S, \\ \rho = e &= \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) + c, \quad \mathbf{V} = \mathbf{1}. \end{aligned} \quad (6)$$

where $\mathbf{k} = (\pi, \pi)$, $\omega = \pi$, $c = 3.0$. Computational domain is $[0, 2] \times [0, 2]$, and periodic boundary condition is applied. Tables 1 show that the proposed methods also maintain the desired-accuracy for the compressible Euler equations.

Double Mach Reflection

This is one of the most well-known test cases for high-resolution schemes. With the computational domain of a tube with a 30 degree ramp, a strong moving shock with $M_s = 10$ impinges on the ramp. AUSMPW+ scheme is used as a numerical flux, and computation is carried out until $t = 0.2$.

Figure 1 shows the comparison of density contours with the second-order FVM-MLP and the DG-MLP methods on triangular grid ($h = 1/200$). The MLP methods successfully provide

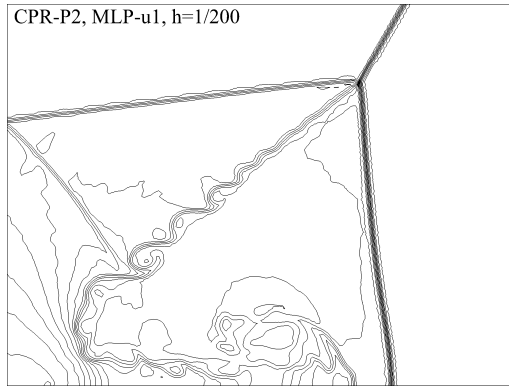


Figure 1. Comparison of density contours around the double Mach stem.

monotonic solutions. The higher-order CPR reconstructions significantly improve the resolution of the shear layer and vortex developed from the shock triple point and the Mach stem.

CONCLUSION

Hierarchical MLP limiting is successfully developed and extended on higher-order CPR method. The proposed method is able to accurately capture complex multi-dimensional compressible flow structures without yielding unwanted oscillations. Various numerical results show the desirable characteristics of the proposed limiting strategy, such as multi-dimensional monotonicity, improved accuracy and efficiency.

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REFERENCES

1. Huynh, H. T., "A Flux Reconstruction Approach to High-order Schemes including Discontinuous Galerkin Methods" , *Proc. of 18th AIAA Computational Fluid Dynamics Conference*, AIAA 2007-4079, June 2007.
2. Wang, Z. J. and Gao, H., "A unifying lifting collocation penalty formulation including the discontinuous galerkin, spectral volume/difference methods for conservation laws on mixed grids" , *Journal of Computational Physics*, Vol 228, 2009, pp. 8161-8186.
3. Park, J. S., Yoon S.-H. and Kim, C., "Multi-dimensional limiting process for hyperbolic conservation laws on unstructured grids", *Journal of Computational Physics*, Vol 229, 2010, pp. 788-812.
4. Park, J. S. and Kim, C., "Higher-order mutli-dimensional limiting strategy for discontinuous Galerkin methods in compressible inviscid and viscous flows", *Computers & Fluids*, submitted.